A dynamic marketing model with best reply and inertia

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Abstract. In this paper we consider a nonlinear discrete-time dynamic model proposed by Farris et al. in the paper "When five is a crowd in the market share attraction model: the dynamic stability of competition" Journal of Research and Management, (2005), as a market share attraction model with two firms that decide marketing efforts over time according to best reply strategies with naïve expectations. The model also considers an adaptive adjustment towards best reply, a form of inertia or anchoring attitude, and we investigate the effects of heterogeneities among firms. A rich scenario of local and global bifurcations is obtained even with just two competing firms, and a comparison is proposed with apparently similar duopoly models based on repeated best reply dynamics with naïve expectations and adaptive adjustment.

Key-words: Market share models; Heterogeneity; Nonlinearity; Symmetry; Stability; Bifurcations.

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1. Introduction

An important stream in the literature on dynamic models in marketing is based on *market share attraction models*, where several firms selling the same good (or more generally homogeneous goods) are competing in a market with a given sales potential, and each firm has to decide its marketing effort in order to maximize (or at least increase) its market share (see e.g. [3], [9], [11], [4], [15]). In this paper we consider the discrete-time dynamic model proposed by Farris et al. [15] where, at each time step t, in order to decide their marketing effort at time $t + 1$, the firms solve an optimization problem to maximize their expected profits. However, their information set is limited as they do not know the effort decisions of their competitors and their are assumed to adopt naïve expectations about competitors' choices, i.e. they guess that marketing efforts will be the same as in the current period. The resulting model, denoted as Best Response with naïve expectations, is well known in the literature since the pioneering work of Cournot (1838) on mathematical modelling of oligopolies (see [12] or any standard textbook on oligopoly modelling). As suggested in [15], as well as by several other authors, see e.g. [23], [7], the awareness of the systematic error in the assumption of naïve expectations, as well as the difficulties to change the marketing policy, may induce the firms to adopt a compromise (a convex combination) between the computed best response and the previously adopted efforts, a form of inertia or anchoring attitude. This leads to the adaptive adjustment model considered in this paper. The analysis of this model given in [15] is mainly focused on the case of n homogeneous firms, i.e. characterized by identical parameters, in order to study the relation between market stability and the number of firms, by taking the number n of firms as a bifurcation parameter in a one-dimensional model that summarizes the common behaviour of the identical firms. In this paper we follow a complementary approach, in the sense that we consider just two firms competing in the same market and we stress the effects of heterogeneities, i.e. how the differences between the parameters, that characterize the dynamic behaviour of the two firms, influence the equilibrium points, their stability and bifurcations, as well as the global dynamic scenarios of the model. A first step has been made towards this direction in the recent paper [8], where the effects of small heterogeneities have been investigated starting from the one-dimensional model analyzed in [15]. However, the rich spectrum of dynamic behaviors that the two-dimensional discrete-time model exhibits under different parameters' constellations is worth to be further investigated, in particular in regions of parameters' space where heterogeneity between the two firms may play an important role. Analytical results can be easily proved for the model obtained under assumptions of some (not all) identical parameters. Moreover, even if numerical explorations are necessary in order to investigate the global properties of the model with arbitrary parameters' values, some general statements can still be given and may stimulate further studies concerning both the economic and the mathematical properties of the model.

The plan of the paper is as follows. In Section 2 the economic model is described and it is written in the form of two-dimensional map with a simple mathematical structure. In Section 3 existence and stability of equilibrium points is proved for the case with identical efforts' effectiveness and different inertia of firms, and some numerical simulations are given to confirm and extend the analytical results. In Section 4 a complete analytical study of the existence of equilibrium points is given for arbitrary values of the parameters, and some numerical simulations are provided in order to characterize the role of marketing efforts on the dynamic properties of the model. Section 5 concludes.

2. The model

Let us consider n firms that sell homogeneous goods in a market with sales potential B (in terms of overall customers' market expenditures, also denoted as *market contribution* MC by some authors, see e.g. [15]), and let $A_i(t)$, $i = 1, ..., n$, denote the attraction of customers to firm i at time period t, where $t \in \mathbb{N}$ denotes an event-driven discrete time variable. The key assumption in marketing literature is that the market share for firm i at time t is given by

$$s_i(t) = \frac{A_i(t)}{\sum_{j=1}^{n} A_j(t)} \tag{1}$$

If x_i denotes marketing spending of firm i, following [11], see also [15], we assume that attraction is given by

$$A_i = a_i x_i^{\beta_i}$$

where the positive constants a_i denote the relative effectiveness of effort expended by the firm i and the parameter β_i denote the elasticity of the attraction of firm (or brand) i with regard to the marketing effort, as $\frac{dA_i}{dx_i}\frac{x_i}{A_i} = \beta_i$. On the basis of these assumptions, the one-period net profit of firm i is given by

$$\Pi_i(t) = Bs_i(t) - x_i(t) = B\frac{a_i x_i^{\beta_i}(t)}{a_i x_i^{\beta_i}(t) + \sum_{j\neq i} a_j x_j^{\beta_j}(t)} - x_i(t) \qquad (2)$$

In [15] the case of unit elasticities $\beta_i = 1$, $i = 1, ..., n$, is considered and at each time t agents are assumed to decide next period spending $x_i(t+1)$ by solving the optimization problem

$$\max_{x_i} \Pi_i^{(e)}(t+1) = \max_{x_i}\left(B\frac{a_i x_i}{a_i x_i + \sum_{j\neq i} a_j x_j^{(e)}(t+1)} - x_i\right)$$

where $\Pi_i^{(e)}(t+1)$ is the expected profit at time $t+1$ and $x_j^{(e)}(t+1)$ represent the expectation of firm i about firm j spending at time $t+1$ on the basis of the information set of firm i at time t. From the first order conditions $\frac{\partial \Pi_i(t+1)}{\partial x_i} = 0$, one gets

$$x_i(t+1) = \sqrt{B\frac{\sum_{j\neq i} a_j x_j^{(e)}(t+1)}{a_i}} - \sum_{j\neq i} a_j x_j^{(e)}(t+1)$$

Assuming naïve expectations

$$x_j^{(e)}(t+1) = x_i(t)$$

the following dynamic model is obtained

$$x_i(t+1) = R_i\left(\sum_{j\neq i} a_j x_j(t)\right) = \sqrt{B\frac{\sum_{j\neq i} a_j x_j(t)}{a_i}} - \sum_{j\neq i} a_j x_j(t) \qquad (3)$$

usually denoted as “Best Response” with naïve expectations. Notice that the parameter B is just a scale parameter, that has no influence on the dynamic properties of the model. So, without loss of generality we shall consider $B = 1$ in the following.

In [15] also a different adjustment process is proposed, known as *adaptive adjustment towards best reply*, see also [23], [19], [7], given by

$$x_i(t+1) = (1-\lambda_i)x_i(t) + \lambda_i R_i\left(\sum_{j\neq i} a_j x_j(t)\right) \qquad i = 1,...,n \tag{4}$$

where the constants $\lambda_i \in [0,1]$ represent the attitude of firm i to adopt the best reply, whereas $(1-\lambda_i)$ is the anchoring attitude to maintain the previous spending decision, i.e. a measure of firms' inertia to modify their marketing efforts. The model (4) is a generalization of (3) because it reduces to it for $\lambda_i = 1$, $i = 1,...,n$, whereas it tends to complete inertia of firm i, i.e. $x_i(t+1) = x_i(t)$, as $\lambda_i \to 0$. It is straightforward to see that for $\lambda_i \neq 0$, $i = 1,...n$, the model with inertia (4) has the same equilibrium points as the model (3). These equilibrium points, being located at the intersections between the two best response functions (3), are Nash equilibria. However the presence of inertia influences their stability properties.

In the following we consider the case of two firms, $n = 2$, with the dynamic variables rescaled as

$$x = a_1 a_2 x_1 \ ; \quad y = a_1 a_2 x_2 \tag{5}$$

so that the dynamic model studied in this paper assumes the form of the map $T : (x, y) \to (x', y')$ with

$$T : \begin{cases} x' = (1-\lambda_1)x + a_2\lambda_1\left(\sqrt{y} - y\right) \\ y' = (1-\lambda_2)y + a_1\lambda_2\left(\sqrt{x} - x\right) \end{cases} \tag{6}$$

where $'$ denotes the unit-time advancement operator, that is, if the right-hand side variables are productions of period t then the left-hand ones represent productions of period $(t+1)$.

The map (6) is defined for nonnegative values of x and y. Starting from a given initial condition $(x_0, y_0) \in \mathbb{R}^2_+$, where $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 | x \geq 0 \text{ and } y \geq 0\}$ denotes the set of nonnegative state variables, the iteration of (6) generates an infinite sequence of states, or a *trajectory*

$$\left\{(x_t, y_t) = T^t(x_0, y_0), t = 1, 2, ...\right\} \tag{7}$$

provided that (x_0, y_0), as well as all its images $T^t(x_0, y_0)$ of any rank t, belong to $\mathbb{R}^2_+$. If $a_i \leq 4$, $i = 1, 2$, then the square $S = [0,1] \times [0,1]$ is a trapping region, i.e. trajectories starting inside S remain in it for each $t \geq 0$. However, feasible (i.e. non-interrupted) trajectories can be even obtained for $a_i > 4$ provided that $\lambda_j a_i \leq 4$, $i = 1, 2$, $j = 1, 2$, $j \neq i$, but this depends on the initial condition, i.e. in this case trajectories starting inside S may be interrupted because of negative values of a dynamic variable. In the following we shall denote unfeasible region the set of points that generate interrupted trajectories and the considered initial conditions will be taken outside such region.

A similar dynamic model has been proposed as a Cournot duopoly model with isoelastic demand and linear cost functions in [23] (see Appendix for details where we show that the duopoly model reduces to the present model for $a_1 a_2 = 1$), see also [1] and [2] for a deeper local and global dynamic analysis of such duopoly model. However, despite such apparently similar form of the map, the dynamic properties of the marketing model (6) reveal to be quite different (we may say even surprisingly different) with respect to the duopoly Cournot model with inertia proposed in [23]. In fact, as shown in the Appendix (see also [23], [1], [2]) such duopoly model has only one nontrivial equilibrium point that may lose stability via a Neimark-Sacker bifurcation, whereas the marketing model (6), as we shall see in the following, may have more than one equilibrium, and different kinds of bifurcations can be observed, such as pitchfork, flip or saddle node bifurcations.

For $\lambda_i \neq 0$ the fixed points of the map (6), obtained by imposing the conditions $x' = x$ and $y' = y$, are the real and non-negative solutions of the algebraic system

$$\begin{aligned} x &= a_2 \left(\sqrt{y} - y \right) \\ y &= a_1 \left(\sqrt{x} - x \right) \end{aligned} \tag{8}$$

It is straightforward to see that $E_0 = (0, 0)$ is always a fixed point and a further real positive solution of (8) always exists, say $E_1 = (x_1, y_1)$ with $x_1 \in (0, 1)$ and $y_1 \in (0, 1)$. Furthermore, as we shall prove in the following, a couple of real positive solutions, say $E_2 = (x_2, y_2)$ and $E_3 = (x_3, y_3)$ may exist according to the parameters' values a_1 and a_2.

Let us notice that, from (2) with the new variables (5) and unit elasticities $\beta_i = 1$, the profits assume the form

$$\Pi_1(x, y) = \frac{a_1 x}{a_1 x + a_2 y} - \frac{x}{a_1 a_2}; \quad \Pi_2(x, y) = \frac{a_2 y}{a_1 x + a_2 y} - \frac{y}{a_1 a_2} \tag{9}$$

and after some trivial algebra the inequality $\Pi_1 > \Pi_2$ becomes

$$G(x,y) = a_1 x^2 + (a_2 - a_1)\, xy - a_2 y^2 - a_1^2 a_2 x + a_1 a_2^2 y < 0 \tag{10}$$

hence the isoprofit curve $G(x,y) = 0$ is an hyperbola with symmetry center $(x_c, y_c) = \left(\frac{a_1 a_2^2 (3a_1 - a_2)}{(a_1+a_2)^2}, \frac{a_1^2 a_2 (3a_2 - a_1)}{(a_1+a_2)^2} \right)$ and asymptotes of slopes a_2 and $-a_1$ respectively. As we shall see, the asymmetric dynamics of the dynamical system (6) may converge to different attractors. According to the region in which such attractors are included one or the other firm may gain higher profits. When several different attractors coexist this crucially depends on initial conditions, so a study of the global structure of the basins of attraction gives information about the firm that prevail in the market, in the sense of gaining higher profits.

Before starting to analyze the local and global dynamic properties of the map (6) it is worth to notice that it is a *noninvertible map*, because if we compute (x, y) in terms of a given (x', y') by solving the system (6) we can get up to four real solutions. i.e. a point can have several rank-1 preimages. Geometrically, the action of a noninvertible map can be expressed by saying that it folds and pleats" the phase space, because distinct points are mapped into the same point. This is equivalently stated by saying that several inverses are defined that unfold" the phase space. For a noninvertible map, the phase space can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages (see e.g. [22]). Generally, for a continuous map, as the point $\mathbf{x}'$ varies in $\mathbb{R}^2$, pairs of preimages appear or disappear as it crosses the boundaries separating different regions. Such boundaries, denoted as LC (from the French Ligne Critique") are defined as sets of points with two merging preimages, located on LC_{-1} (following the notations of [18], [22]). For a differentiable noninvertible map of the plane, the set LC_{-1} is the set where the Jacobian determinant vanishes:

$$LC_{-1} = \left\{ (x,y) \in \mathbb{R}^2 |\, \det \mathbf{J} = 0 \right\} \tag{11}$$

(see again [18], [22]) where, in our case,

$$\boldsymbol{J}(x,y) = \begin{pmatrix} 1 - \lambda_1 & \lambda_1 a_2 \left(\dfrac{1}{2\sqrt{y}} - 1 \right) \\ \lambda_2 a_1 \left(\dfrac{1}{2\sqrt{x}} - 1 \right) & 1 - \lambda_2 \end{pmatrix} \tag{12}$$

So, we get the following equation for LC_{-1}

$$\left(\frac{1}{2\sqrt{x}}-1\right)\left(\frac{1}{2\sqrt{y}}-1\right)=\frac{(1-\lambda_1)(1-\lambda_2)}{a_1a_2\lambda_1\lambda_2} \tag{13}$$

formed by the union of two disjoint branches, say $LC_{-1}=LC_{-1}^{(a)}\cup LC_{-1}^{(b)}$. Also $LC=T(LC_{-1})$ is the union of two branches: $LC^{(a)}=T(LC_{-1}^{(a)})$ and $LC^{(b)}=T(LC_{-1}^{(b)})$. The branch $LC^{(a)}$ separates the regions Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages. The other branch, $LC^{(b)}$, separates the region Z_2 from Z_4, whose points have four distinct preimages.

3. The case $a_1 = a_2$

In this section we consider the case of identical effort effectiveness

$$a_1=a_2=a \tag{14}$$

so that the firms can only differ with respect to their inertia, expressed by the parameters λ_1 and λ_2. Under this assumption the system (8) can be analytically solved, and besides E_0 the following fixed points are obtained:

$$E_1=\left(\frac{a^2}{(1+a)^2},\frac{a^2}{(1+a)^2}\right)$$

located on the diagonal $\Delta=\{(x,y)\in\mathbb{R}^2|x=y\}$, and for $a\geq 3$ two further fixed points in symmetric positions with respect to Δ

$$E_2=\frac{a^2}{2\,(a-1)^2\,(a+1)}\left(a-1+\sqrt{(a+1)\,(a-3)},\ a-1-\sqrt{(a+1)\,(a-3)}\right)$$
$$E_3=\frac{a^2}{2\,(a-1)^2\,(a+1)}\left(a-1-\sqrt{(a+1)\,(a-3)},\ a-1+\sqrt{(a+1)\,(a-3)}\right)$$

The local stability of the fixed points is described by the following statement:

Proposition 1. *The fixed point $E_1\in\Delta$ is locally asymptotically stable for $a<a_p=3$. At $a=a_p$ a pitchfork bifurcation occurs at which the two fixed*

points E_2 and E_3 are created and are stable nodes just after the bifurcation. The fixed point $E_1 \in \Delta$ also undergoes a flip bifurcation at $a = a_f \geq 3$, being

$$a_f = 1 + 2\sqrt{1 + 2\frac{2 - \lambda_1 - \lambda_2}{\lambda_1\lambda_2}} \tag{15}$$

at which a cycle of period 2 is created along Δ. For $a > a_h$, with

$$\begin{aligned} a_h = 1 + &\sqrt[3]{2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + 2\sqrt{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 - \frac{16}{27}}} + \\ &+ \sqrt[3]{2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - 2\sqrt{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 - \frac{16}{27}}} \end{aligned} \tag{16}$$

the two fixed points E_2 and E_3 lose stability via a Neimark-Sacker bifurcation.

Proof. The Jacobian matrix (12), with $a_1 = a_2 = a$, computed at the fixed point E_1 becomes

$$\boldsymbol{J}(E_1) = \begin{pmatrix} 1 - \lambda_1 & \lambda_1 \dfrac{1-a}{2} \\ \lambda_2 \dfrac{1-a}{2} & 1 - \lambda_2 \end{pmatrix}$$

From the characteristic equation

$$P(z) = z^2 - Tr \cdot z + Det = 0 \ ,$$

where $Tr = 2 - \lambda_1 - \lambda_2$ and $Det = (1 - \lambda_1)(1 - \lambda_2) - \lambda_1\lambda_2 (1 - a)^2 /4$ are the trace and the determinant of $\boldsymbol{J}(E_1)$ respectively, a sufficient condition for the stability is expressed by the following system of inequalities (known as Schur or Jury's conditions, see e.g. [16], [14], [21])

$$\begin{aligned} P(1) = 1 - Tr + Det > 0 \\ P(-1) = 1 + Tr + Det > 0 \\ 1 - Det > 0 \end{aligned} \tag{17}$$

which are equivalent to state that the two eigenvalues are located inside the unit circle of the complex plane. In our case

$$\begin{aligned}
P(1) &= \lambda_1\lambda_2\left(1-\frac{(1-a)^2}{4}\right) > 0 \text{ for } a<3 \\
P(-1) &> 0 \;\text{ for } a<a_f \\
1-Det &= \lambda_1+\lambda_2-\lambda_1\lambda_2+\lambda_1\lambda_2\frac{(1-a)^2}{4} > 0\; \forall a,\; \lambda_i\in[0,1]
\end{aligned}$$

where

$$a_f = 1+2\sqrt{1+2\frac{2-\lambda_1-\lambda_2}{\lambda_1\lambda_2}}$$

For $a>3$, the Jacobian matrix (12) computed at the two symmetric fixed points $E_2=(x_2,y_2)$ and $E_3=(y_2,x_2)$ is the same, and has has complex conjugate eigenvalues

$$z_{1,2}=\frac{1}{2}(2-\lambda_1-\lambda_2)\pm\sqrt{(\lambda_1-\lambda_2)^2+\lambda_1\lambda_2\left(-a^3+3a^2+a+1\right)}$$

hence they are on the unit circle for $a=a_h$ at which $\det(J(E_2))=z_1z_2=|z_1|^2=1$, and exit the unit circle, i.e. $z_1z_2=|z_1|^2>1$, for $a>a_h$. This completes the proof. □

Notice that $a_f=a_p=3$ if and only if $\lambda_1=\lambda_2=1$, i.e. the two firms exhibit no inertia. In this case, at $a=3$ a degenerate bifurcation of codimension 2 occurs at which the fixed point $E_1\in\Delta$ is transformed from stable node into unstable node and, simultaneously a pair of stable fixed points are created in symmetric positions with respect to the diagonal Δ and a stable cycle of period 2 is created along the invariant diagonal Δ, as well. So, after this bifurcation, three coexisting attractors are present, each with its own basin of attraction (see fig. 1a, obtained with parameters $\lambda_1=\lambda_2=1$ and $a=3.03>a_p$). The particular "rectangular shaped" structure of the basins is a consequence of the fact that for $\lambda_1=\lambda_2=1$ the map (6) assumes the form $(x',y')=T(x,y)=(f(y),g(x))$, hence it maps horizontal lines into vertical lines and vice-versa. The properties of attractors and basins of these maps, characterized by the fact that $T^2(x,y)=(f(g(x)),g(f(y)))$ is uncoupled, are studied in [5].

Instead, if $\lambda_1 = \lambda_2 = \lambda < 1$, i.e. the firms are homogeneous with the same degree of inertia, the structure of attractors and basins is still symmetric with respect to the invariant diagonal Δ, but with a more smooth shape. Moreover, from (15) it follows that $a_f > 3$, i.e. $a_f > a_p$. In this case, as the parameter a crosses the bifurcation value a_p, the fixed point E_1 is transformed from a stable node into a saddle point, with local stable set along the invariant diagonal Δ, which becomes a boundary that separates the basin of the two newly born stable nodes E_2 and E_3 (see the situation represented in fig. 1b, obtained with $\lambda = 0.9$ and $a = 3.3$, i.e. $3 = a_p < a < a_f = 3.\overline{4}$). If the parameter a is further increased across the bifurcation value a_f, at which the flip bifurcation occurs, the saddle point E_1 gives rise to a saddle cycle of period 2 with periodic points along Δ. As a is further increased this saddle cycle gains transverse stability via a subcritical flip bifurcation and this leads to the dynamic scenario represented in fig. 1c, obtained with $\lambda = 0.9$ and $a = 3.7$. So, at this stage we have again three coexisting attractors, one along Δ (a cycle of period 2) and two steady states in symmetric positions with respect to Δ, each with its own basin of attraction, represented by different gray shades in fig.1. Notice that the basins are non connected sets, a phenomenon that can only be observed with noninvertible maps.

With the same parameters λ_i, $i = 1, 2$, the Neimark-Sacker bifurcation occurs at $a_h \simeq 3.71$, according to (16) of Proposition 1, and the two fixed points E_2 and E_3 become unstable focuses, and they are surrounded by stable closed invariant curves along which quasi-periodic motion occurs. This numerically proves that the Neimark-Sacker bifurcation is supercritical (see fig. 1d).

It is worth to notice that, from (16) we have $a_h \geq 3.645$, as $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \geq 2$ and being a_h increasing with $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)$.

Up to now we have only been concerned with the time evolution of marketing efforts $x_1(t)$ and $x_2(t)$ of the two firms, related to the rescaled dynamic variables $x(t)$ and $y(t)$ by (5). However, it is also important to consider the time evolution of profits as well. If $a_1 = a_2 = a$ then the condition (10), equivalent to $\Pi_1 > \Pi_2$, becomes

$$(x - y)\left(x + y - a^2\right) < 0, \tag{19}$$

i.e. the hyperbola $G(x, y) = 0$ degenerates into the pair of orthogonal lines

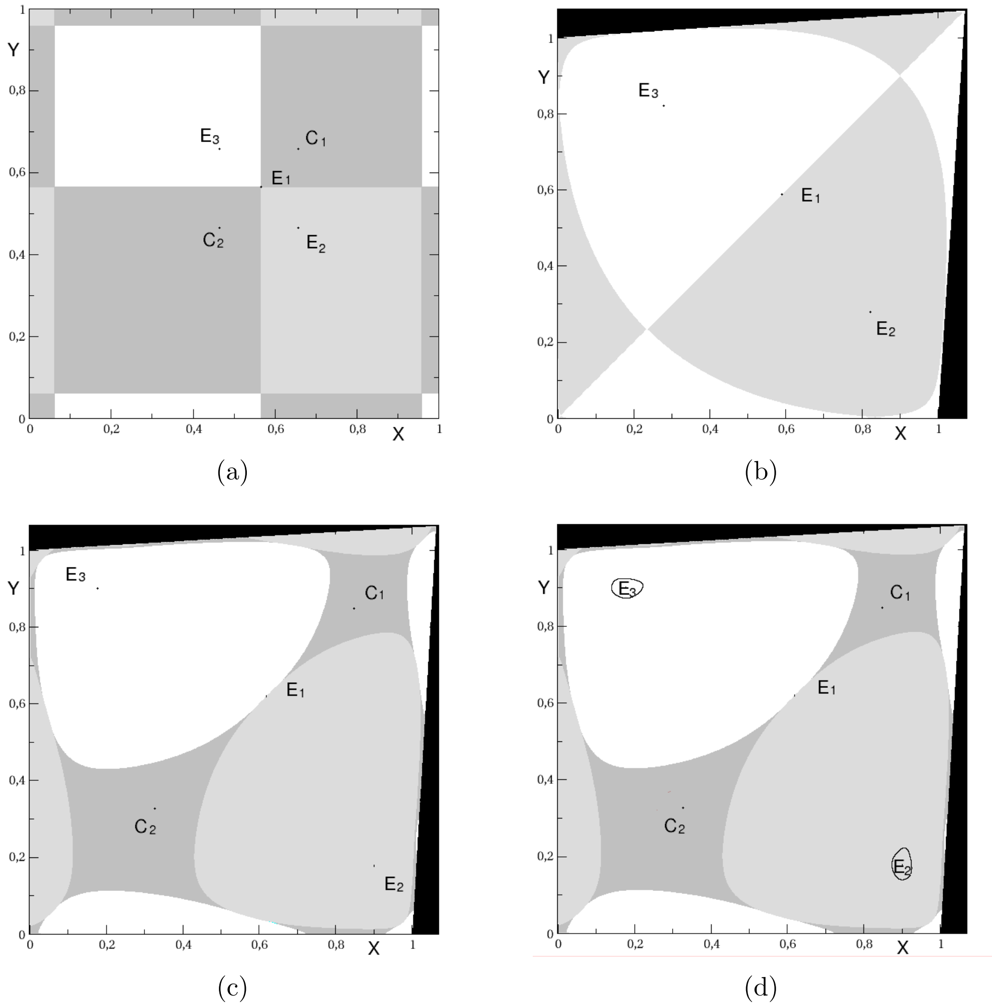


Figure 1: Panel (a): $\lambda_1 = \lambda_2 = 1$ and $a = 3.03 > a_p$, just after the codimension 2 bifurcation at which simultaneous flip and pitchfork bifurcations occur. Panel (b): $\lambda_1 = \lambda_2 = 0.9$ and $a = 3.3$, just after the pitchfork bifurcation: $3 = a_p < a < a_f = 3.\overline{4}$. Panel (c): $\lambda_1 = \lambda_2 = 0.9$ and $a = 3.7$, just after the subcritcal flip bifurcation of the cycle of period 2 along the diagonal. Panel (d): $\lambda_1 = \lambda_2 = 0.9$ and $a = 3.72$, just after the Neimark-Sacker bifurcation of equilibrium points E_2, E_3, being $a_h = 3.71$. The different gray shades represent basins of attraction, wherease the black region represents the initial conditions that generate unfeasible trajectories, i.e. involving negative values of the dynamic variables.

$y = x$ and $y = -x + a^2$, crossing at $(x_c, y_c) = \left(\frac{a^2}{2}, \frac{a^2}{2}\right)$, that divide the phase space into four sectors such that $\Pi_1 > \Pi_2$ in the two sectors above both lines and below both lines, and $\Pi_1 < \Pi_2$ in the other two sectors. At the equilibrium E_1, being $x = y$, the two firms have the same profits, whereas for

$a > 3$ at the two equilibrium points E_2 and E_3 the inequalities $\Pi_1 > \Pi_2$ and $\Pi_1 < \Pi_2$ hold respectively. In fact, at E_2 and E_3 we have $x + y = \frac{a^2}{a^2-1} < a^2$, so it is easy to realize that at the equilibrium E_2, characterized by $x > y$ (i.e. firm 1 spends more in marketing efforts) we have $\Pi_1 > \Pi_2$, i.e. firm 1 makes more profits, and vice-versa in the equilibrium E_3. Similar situations are obtained in the case of more complex attractors, located around these equilibria when they are unstable, in the sense that $\Pi_1 > \Pi_2$ when the dynamics occur along attractors located in the region with $x > y$ (i.e. below the diagonal Δ) and vice-versa for attractors in the region $x < y$ (above the diagonal Δ). In fact, for $a > 3$ the line $x + y = a^2$ is outside the region where the asymptotic dynamics of the model take place, so that only the first factor in (19) determines the difference between the two profits. Instead, in the case of attractors that include both regions with $x > y$ and $x < y$, time periods with $\Pi_1 > \Pi_2$ alternate with time periods with $\Pi_1 < \Pi_2$.

This occurs, for example, in the case of homogeneous firms when the coexisting chaotic attractors created around the unstable equilibrium points E_2 and E_3 merge and form a unique large chaotic attractor, as shown in fig. 2. In the left panel (fig. 2a) when the asymptotic motion occurs along the attractor surrounding the equilibrium E_2 then firm 1 makes more profits, and vice-versa for the trajectories converging to the upper attractor around E_3. After the contact between the two attractors along the diagonal Δ, occurring for increasing values of a, the motion along the unique large attractor is characterized by alternating time periods with higher profit for one firm or the other one.

However, as stated in the Introduction, in this paper we are mainly interested to study the cases of heterogeneous firms. We start by discussing the effects of different inertia, i.e. $\lambda_1 \neq \lambda_2$. In this case the diagonal is no longer invariant, i.e. $x = y$ in (6) does not imply $x' = y'$. This has a remarkable effect on the shape of the basins, as shown in fig. 3a, obtained with $\lambda_1 = 0.5$ and $\lambda_2 = 0.7$ (i.e. firm 1 has more inertia in revising efforts than firm 2) and $a = 4$, i.e. just after the Neimark-Sacker bifurcation of E_2 and E_3. In this case the stable set of the saddle point E_1, which constitutes the boundary that separates the two basins of attraction, is folded so that the basin of the attractor below the diagonal, where firm 1 makes more profits, is smaller. Notice that firm 1 is the one with more inertia. In the situation shown in fig. 3b, obtained again for $a = 4$ but with $\lambda_1 = 0.7$ and $\lambda_2 = 1$ (firm 2

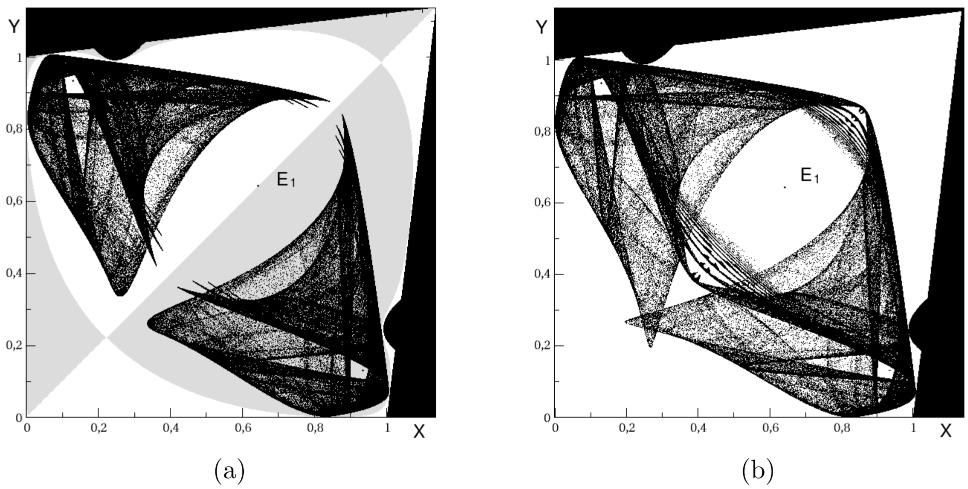
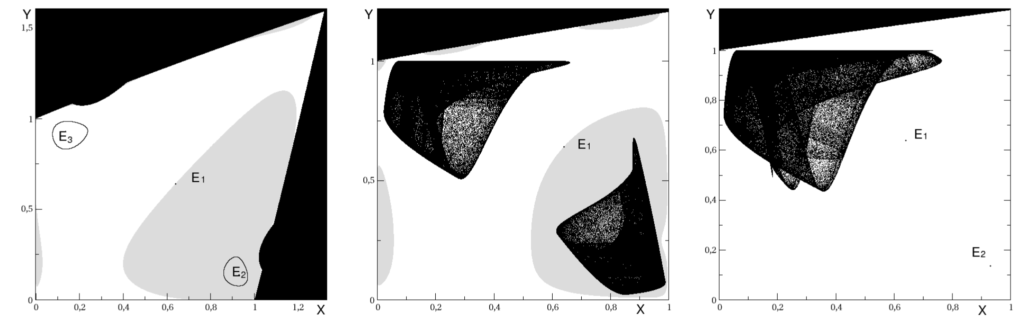


Figure 2: Panel (a): for $\lambda_1 = \lambda_2 = 0.8$ and $a = 4.05$ two chaotic attractors coexist each with its own basin of attraction. Panel (b): for $\lambda_1 = \lambda_2 = 0.8$ and $a = 4.0565$ a unique symmetric chaotic attractor exists.

exhibit no inertia, i.e. it moves directly to the computed best reply marketing effort) the two attractors become chaotic, and the basin of attraction of the one surrounding E_2 (characterized by higher profits of firm 1) shrinks further. As it can be seen in fig. 3b, the chaotic attractor around E_2 is quite close to the basin boundary. Indeed, if λ_1 is slightly increased, then such a contact occurs, leading to the disappearance of the chaotic attractor around E_2, a global (or contact) bifurcation known as "final bifurcation" (see [22]) or "boundary crisis" (see [17]). After this contact bifurcation the generic trajectory goes to the chaotic attractor around E_3 (see fig. 3c, obtained with $\lambda_1 = 0.75$, $\lambda_2 = 1$ and $a = 4$). The former chaotic attractor around E_2 is transformed into a chaotic repellor, formed by the dense and unstable set of periodic points that constituted the skeleton of the just disappeared attractor: such chaotic repellor is often denoted as the "ghost" of the attractor, and its presence gives rise to long chaotic transients along the former chaotic attractor before the generic trajectory reaches the other attracting set (now globally attracting).

In the dynamic scenarios shown in figures 3a,b, as well as those in fig.1, the feature of non connected basins of attraction is quite evident, a property specific to noninvertible maps which is very important in applications, as it



(a) (b) (c)

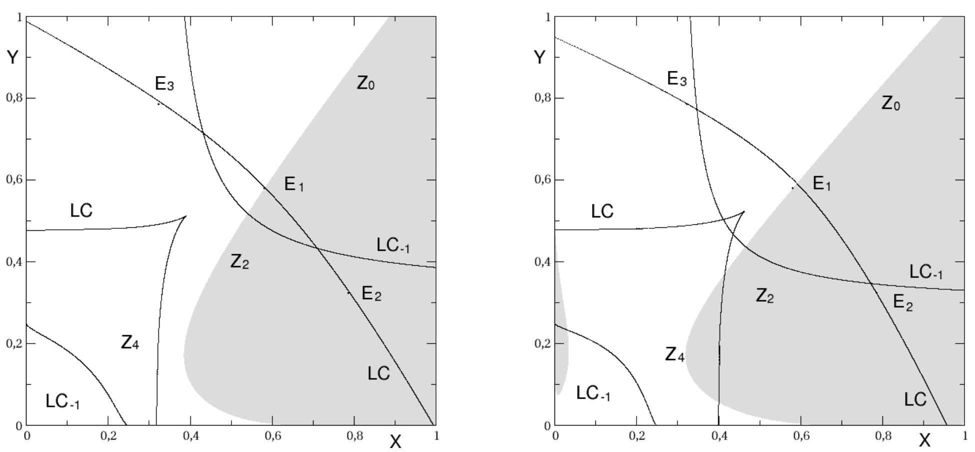
Figure 3: Panel (a): $\lambda_1 = 0.5$, $\lambda_2 = 0.7$ and $a = 4$. Panel (b): $\lambda_1 = 0.7$, $\lambda_2 = 1$ and $a = 4$. Panel (c): $\lambda_1 = 0.75$, $\lambda_2 = 1$ and $a = 4$. The white and the light gray in (a) and (b) represent the basins of the two coexisting attractors, whereas in (c) the generic trajectory starting from an initial condition in the white region enters the unique (chaotic) global attractor.

has a strong influence on path dependence. This property is now well known, after the books [18] and [22], as well as many papers and books dealing with applications, see e.g. [25], [4], [6], where the transition between connected and non connected basins is explained in terms of the unfolding action of critical curves. We give here just a short qualitative explanation, based on the fact that in the context of noninvertible maps it is useful to define the *immediate basin* $\mathcal{B}_0(A)$, of an attracting set A, as the widest connected component of the basin which contains A. Then the total basin can be expressed as

$$\mathcal{B}\,(A) = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{B}_0(A))$$

where $T^{-n}(x)$ represents the set of all the rank-n preimages of x, i.e. the set of points which are mapped in x after n iterations of the map T. The backward iteration of a noninvertible map *repeatedly unfolds* the phase space, and this implies that the basins may be non-connected, i.e. formed by several disjoint portions. As recalled at the end of Section 2, each branch of critical curve $LC = T(LC_{-1})$ separates regions of the phase plane characterized by different numbers of preimages. So, if a portion of a basin, after a contact with a critical curve, enters the region Z_k characterized by a higher number of preimages, then the extra preimages created after the contact may form a nonconnected portion of the basin.

On the basis of these arguments, in fig. 4 we show how the creation of a nonconnected portion of a basin for the map (6) can be explained in terms of a contact between the immediate basin and a branch of critical curve $LC = T(LC_{-1})$, where LC_{-1} is given by (13). In fig. 4a, obtained for $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$, with $a = 3.2$, the two equilibria E_2 and E_3 are stable focuses, with the stable set of E_1 that separates the two basins, represented by light gray and white respectively. When λ_1 is increased from 0.4 to 0.5 the boundary of the light gray basin of E_2 has a contact with LC and a portion of it enters the region Z_4. This gives rise to the creation of a new non connected portion of the same basin, formed by the union of the two further preimages merging along LC_{-1}, of the portion of the basin that entered the region Z_4 after the contact.



(a) (b)

Figure 4: Panel (a): $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, $a = 3.2$. Panel (b): $\lambda_1 = 0.5$ with the other parameters at the same value as in (a).

To sum up, by tuning the different values of the inertia parameters λ_1 and λ_2, with a fixed common value of marketing effectiveness a, both the kind of attractors and the structure of the basins can be strongly influenced. In the next section we consider the further source of heterogeneity given by different values of effort effectiveness a_1 and a_2.

4. The general case

In this section we relax the assumption (14) of identical effort effectiveness, so that the two firms can differ both for their effort effectiveness and for their inertia.

Unfortunately the simple analytical expression of the fixed points obtained in the previous section is lost, however we can give the following general result concerning their existence.

Proposition 2. *Besides* $E_0 = (0,0)$ *a non vanishing fixed point always exists in the region* $S = (0,1) \times (0,1)$. *If* $a_1 a_2 \neq 1$ *then two further distinct fixed points exist in the region* S *if the following inequality holds*

$$D(a_1, a_2) = \frac{a_1^2 a_2^4}{108\left(1 - a_1 a_2\right)^6}\left[27 + a_1 a_2\left(4a_1 + 4a_2 - 18\right) - a_1^2 a_2^2\right] < 0 \quad (20)$$

and if $D(a_1, a_2) = 0$ *these two further fixed points are merging, i.e. there are two real coincident solutions of (8). In the particular case* $a_1 a_2 = 1$ *the unique fixed point* $E = \left(\frac{1}{(a_1+1)^2}, \frac{1}{(a_2+1)^2}\right)$ *is get.*

Proof. The algebraic system (8), whose solutions are the fixed points of the map (6), can be rewritten as

$$\begin{cases} \zeta = \frac{1-a_1a_2}{a_2}\eta^2 + a_1\eta \\ \eta\left[\frac{(1-a_1a_2)^2}{a_1a_2}\eta^3 + 2\left(1-a_1a_2\right)\eta^2 + a_2\left(a_1+1\right)\eta - a_2\right] = 0 \end{cases} \quad (21)$$

where $\eta = \sqrt{x}$ and $\zeta = \sqrt{y}$. The roots of the third degree polynomial inside square brackets can be found by Cardano's formula. In fact it can be written as

$$\xi^3 + p\xi + q = 0$$

with $\xi = \eta + \frac{2a_1a_2}{3(1-a_1a_2)}$ and coefficients

$$p = a_1 a_2^2 \frac{3 - a_1}{3\left(1 - a_1 a_2\right)^2}\ , \quad q = \frac{a_1 a_2}{27\left(1 - a_1 a_2\right)^3}\left(-27a_2 + 9a_1 a_2^2 - 2a_1^2 a_2^2\right)$$

and the condition for the existance of three real solutions is given by $D = \frac{q^2}{4} + \frac{p^3}{27} < 0$ (see e.g. [24]), that is readily transformed in the form (20). In particular, if $a_2 = 1/a_1$ then the system (21)

$$\begin{cases} \zeta = a_1\eta \\ \frac{\eta}{a_1}\left[\left(a_1+1\right)\eta - 1\right] = 0 \end{cases}$$

from which the unique nonvanishing solution is $\eta = \frac{1}{a_1+1}$, $\zeta = \frac{a_1}{a_1+1} = \frac{1}{a_2+1}$, and consequently $x = \eta^2 = \frac{1}{(a_1+1)^2}$, $x = \zeta^2 = \frac{1}{(a_2+1)^2}$.

Finally, the fact that the real solutions (x, y) of (8) are inside the region S follows easily from direct inspection of (8). In fact, if $y < 0$ then from the second equation x cannot assume a real value, and analogously if $x < 0$ then y cannot be real; if $y > 1$, then from the first equation $x < 0$, which is a contradiction according to the argument given above.□

The graph of $D(a_1, a_2) = 0$ in the parameters' plane (a_1, a_2), shown in fig. 5, is symmetric with respect to the diagonal $a_1 = a_2$ as $D(a_1, a_2) = D(a_2, a_1)$, and is formed by two smooth branches joining at the cusp point located in $a_1 = a_2 = 3$. If the two parameters are below the line $a_1 + a_2 = 6$ then only one equilibrium can exist.

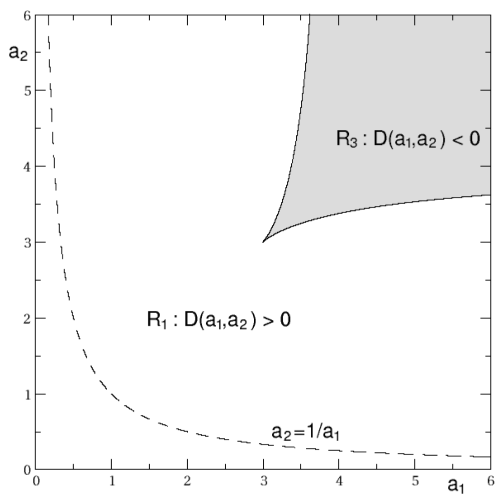


Figure 5: In the parameters' plane (a_1, a_2) the region R_1 (white), where one non trivial equilibrium point exists, and the region R_3 (light gray), where three non trivial equilibrium points exist, are shown. R_1 and R_3 are bounded by the curve $D(a_2, a_2) = 0$. The dotted line is the curve $a_2 = a_1^{-1}$.

Notice that the particular case $a_1 a_2 = 1$ corresponds to the duopoly model with isoelastic demand proposed in [23] (see the Appendix). This model is obtained only in a subset of zero measure in the parameters' plane (a_1, a_2), represented by the dotted curve in fig. 5, which is entirely included inside the region R_1 with only one equilibrium.

If the values of the parameters a_1 and a_2 are allowed to change in the portion of the parameters' plane with $a_1 + a_2 > 6$ a saddle-node bifurcation occurs when a smooth branch of the curve $D = 0$ is crossed, i.e. two equilibrium points are created as the branch is crossed towards the region R_3 bounded by the two branches. Whenever the parameters (a_1, a_2) are located along a branch of $D = 0$ two merging equilibria exist, that split inside the region characterized by $D(a_1, a_2) < 0$ (denoted by R_3 in fig. 5) giving rise to a stable node and a saddle point, whose stable set represents the boundary of the basin of the stable node. Instead, at the cusp point three merging equilibria are obtained, and if the parameters (a_1, a_2) enter region R_3 across the cusp point $(3, 3)$ then a pitchfork bifurcation occurs according to Proposition 1. Three bifurcation diagrams are shown in fig. 6, obtained when the parameters (a_1, a_2) are varied along the paths shown in the small pictures reported inside. As it can be seen in panels (a) and (b) the saddle-node bifurcations, at which pairs of equilibrium points are created or destroyed, give rise to typical hysteresis effects. Instead, the bifurcation path shown in panel (c), across the cusp point, gives rise to the typical pitchfork bifurcation already discussed in the previous section.

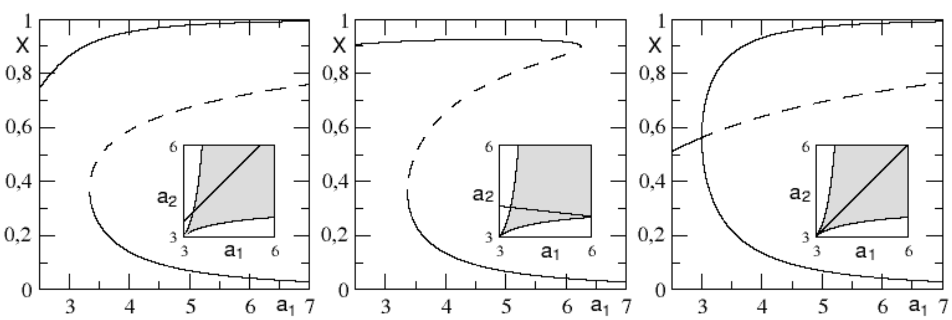


Figure 6: Three bifurcation diagrams when the parameters (a_1, a_2) are varied along the paths shown in the small pictures reported inside.

We end this section by some numerical explorations in order to discuss the role of the parameters a_1 and a_2 on the dynamic scenarios of the model. Let us first consider a sequence of numerical simulations obtained with identical inertia parameters $\lambda_1 = \lambda_2 = 0.8$, starting from a simple dynamic situation, obtained for $a_1 = 3.5$ and $a_2 = 3$, characterized by a unique globally stable equilibrium, denoted by $E_3 = (0.196, 0.864)$ in fig. 7a. As a_2 is increased a couple of further equilibrium points, say E_1 and E_2, is created through a saddle-node bifurcation, the stable set of the saddle E_1 being the boundary that separates the basin of the two attracting equilibria E_2 and E_3. The symmetric case with $a_1 = a_2 = 3.5$ is shown in fig. 7b. As the parameter a_2 is further increased, the equilibrium point E_2 loses its stability via a supercritical Neimark-Sacker bifurcation at which a stable closed invariant curve is created, which is transformed into a chaotic attractor as a_2 increases more and more, see fig. 7c obtained with $a_2 = 4$. If a_2 is further increased then the chaotic attractor enlarges until it has a contact with the boundary that separates its basin with the set of unfeasible trajectories, i.e. the set of points that generate trajectories involving negative values, existing outside the phase space $S = [0,1] \times [0,1]$. At this contact a final bifurcation (or boundary crisis) occurs and, as already described in the previous section, after this contact the generic trajectory starting outside the basin of E_3, is an interrupted path leading to negative values (one firm stops marketing efforts, i.e. it exits the market) as shown in fig. 7d. To sum up, two contrasting effects are obtained by increasing the effort effectiveness a_2 of firm 2: it first leads to the creation of a second attractor and then it leads to its destruction.

A similar effect is obtained when different levels of inertia are considered, like in the sequence of numerical simulations shown in fig. 8 obtained with $\lambda_1 = 0.9$ and $\lambda_2 = 0.7$. In this case the initial dynamic scenario, obtained for $a_1 = 4$ and $a_2 = 3$, is given by the presence of a unique equilibrium, say $E_3 = (0.12, 0.91)$, which is an unstable focus surrounded by a stable periodic cycle of period 5 (five periodic points located along an invariant closed curve) which is a global attractor, i.e. it attracts the generic initial condition in the square $S = [0,1] \times [0,1]$. As a_2 increases, a saddle-node bifurcation occurs at which the two equilibrium points E_1 (saddle) and E_2 (stable node) are created, as shown in fig. 8b obtained with $a_2 = 3.4$. Just after the saddle-node bifurcation the basin of E_2 is very small, being it bounded, as usual, by the stable set of E_1. However this basin, or the basin of the attractor around the originary fixed point, enlarges more and more when a_2 is further increased, as shown in fig. 8c obtained for $a_2 = 3.8$. This is a remarkable global phenomenon, that cannot be revealed through a linear approximations around the attracting sets, and can generally be detected by computer-aided analysis. Also in this case, like in the sequence of numerical simulations analyzed in fig. 7, as a_2 is further increased the attractor around the equilibrium E_2 becomes chaotic, and its size increases until its has a contact with the basin of unfeasible trajectories at which it disappears, i.e.

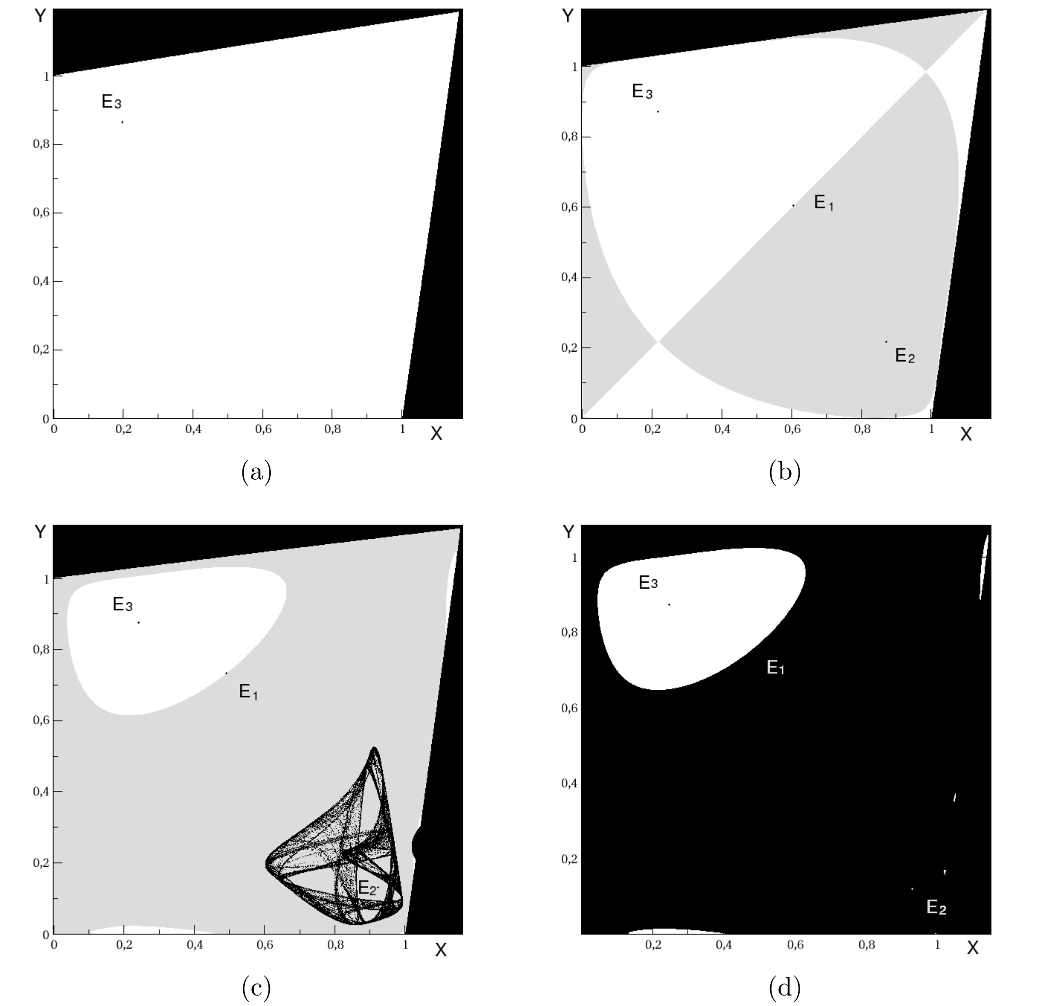


Figure 7: $\lambda_1 = \lambda_2 = 0.8$, $a_1 = 3.5$. In panel (a) $a_2 = 3$; in (b) $a_2 = 3.5$, in (c) $a_2 = 4$; in (d) $a_2 = 4.1$. The white region is the basins of the upper attractor, the light gray region is the basin of the lower attractor, whereas the black region represents the set of points that generate unfeasible trajectories (i.e. entering the negative orthants).

it is transformed into a chaotic repellor that influences the chaotic transient before the trajectories are interrupted.

5. Conclusions

In this paper we have considered the market share attraction model proposed in [15] with just two firms competing in the market that decide their

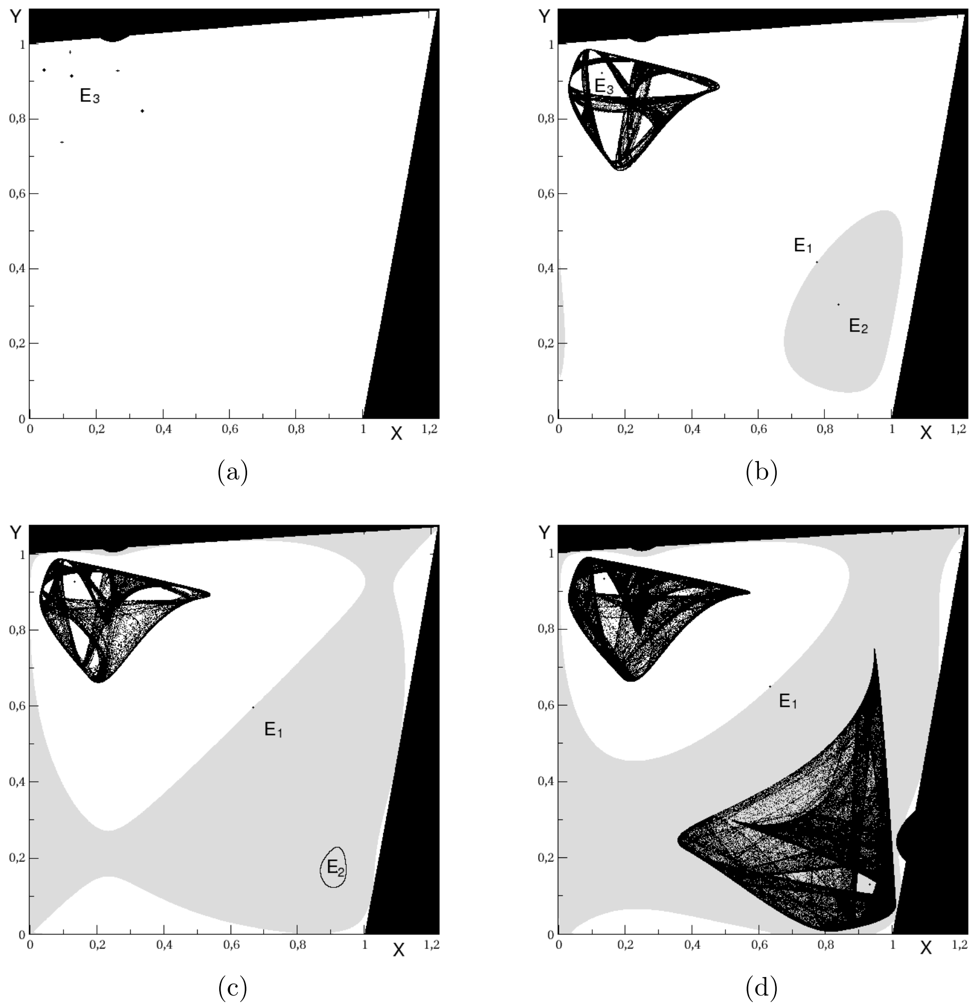


Figure 8: $\lambda_1 = 0.9$, $\lambda_2 = 0.7$, $a_1 = 4$. In panel (a) $a_2 = 3$; in (b) $a_2 = 3.4$; in (c) $a_2 = 3.8$; in (d) $a_2 = 4.05$. The white region is the basins of the upper attractor, the light gray region is the basin of the lower attractor, whereas the black region represents the set of points that generate unfeasible trajectories (i.e. entering the negative orthants

own marketing efforts according to best reply with naïve expectations, and we have stressed the effects of heterogeneities between firms. Through analytical results and numerical analysis, the role of heterogeneities on the existence, stability and structure of the basins of attraction have been stressed. An analytic study of the existence and stability properties of equilibrium points has been given, showing how local and global bifurcations are influenced by differences in the parameters measuring inertia (or prudence) of firms, as well as differences in the relative effectiveness of effort. In particular, it has been shown that increasing inertia often leads, as expected, to more stability of the steady states. However the basin of attraction of the attractor where the profit of the firm with increasing inertia is higher shrinks, up to causing the disappearance of the attractor through final bifurcations (or boundary crisis) Similar phenomena occurs when the parameters that represent marketing efforts effectiveness are increased. These parameters give rise to hysteresis phenomena due to the occurrence of saddle node bifurcations at which couples of steady states are created or destroyed. Moreover, a gradual increase of one of these effort efficiency parameters first cause the creation of a coexisting attractors, and then its disappearance, when it becomes too large, due to a contact bifurcation with the set of points that generate unfeasible trajectories.

The global bifurcations leading to the creation of non connected basins of attraction are discussed, and their meaning in terms of strong path dependence is stressed. The role of heterogeneities has been stressed both in stability properties and in the structure of the basins, as well as the relations between symmetry properties and bifurcation scenarios. A global study of the basins' structure is crucial to forecast which firm will prevale in the market in the sense of gaining higher profits. In fact, the dynamic complexities shown in our analysis include coexistence of attractors of different kinds (steady states, periodic cycles, quasi-periodic and chaotic motions) on which firms have different profits in the long run, with topologically complicated structures of the basins, due to noninvertibility of the discrete dynamical system. This gave us the possibility to stress how the discrete dynamical system considered is quite interesting even from the mathematical point of view. Moreover, the comparison with the (apparently) very similar model proposed in [23] as a discrete-time model of a duopoly game with isoelastic demand function, allowed us to stress that the latter is a quite particular subclass of the marketing model considered in this paper, and consequently the dynamic scenarios analyzed in the marketing model considered in this paper are much richer than the ones observed in the literature for that duopoly game.

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6. Appendix

In this section, just for comparison and in order to make the paper more self-consistent, we recall the map of the Cournot duopoly model proposed in [23] and its main properties.

The map assumes the form

$$\begin{aligned} q_1' &= (1-\lambda_1)\, q_1 + \lambda_1 \left[\sqrt{\frac{q_2}{c_1}} - q_2 \right] \\ y' &= (1-\lambda_2)\, q_2 + \lambda_2 \left[\sqrt{\frac{q_1}{c_2}} - q_1 \right] \end{aligned} \tag{22}$$

where also in this case the parameters $\lambda_i \in [0,1]$ represent the attitude of firm i to adopt the best reply, hence $(1-\lambda_i)$ represents the inertia of firm i, $i = 1,2$, and $c_i > 0$, $i = 1,2$, represent unitary costs. The fixed points, nonnegative solutions of the algebraic system

$$\begin{aligned} q_1 &= \sqrt{\tfrac{q_2}{c_1}} - q_2 \\ q_2 &= \sqrt{\tfrac{q_1}{c_2}} - q_1 \end{aligned}$$

are the trivial one $E_0 = (0,0)$ and the unique positive solution $E_1 = \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2} \right)$. The Jacobian matrix, computed at the fixed point E_1, becomes

$$\boldsymbol{J}(E_1) = \begin{pmatrix} 1-\lambda_1 & \lambda_1 \left(\dfrac{c_1+c_2}{2c_1} - 1 \right) \\ \lambda_2 \left(\dfrac{c_1+c_2}{2c_2} - 1 \right) & 1-\lambda_2 \end{pmatrix}$$

The coefficients of the characteristic equation

$$P(z) = z^2 - Tr \cdot z + Det = 0 \ ,$$

are $Tr = 2-\lambda_1-\lambda_2$ and $Det = (1-\lambda_1)(1-\lambda_2)-\lambda_1\lambda_2\left(\frac{a_1+a_2}{2a_1}-1\right)\frac{a_1+a_2}{2a_2}-1$, hence the Schur conditions for stability become

$$\begin{aligned}
P(1) &= \frac{\lambda_1\lambda_2\,(c_1+c_2)^2}{4c_1c_2} > 0 \text{ for each } c_i > 0,\ \lambda_i \in (0,1] \\
P(-1) &= 8c_1c_2\,(2-\lambda_1-\lambda_2) + \lambda_1\lambda_2\,(c_1+c_2)^2 > 0 \text{ for each } c_i > 0,\ \lambda_i \in (0,1] \\
1 - Det &> 0 \text{ if } \frac{(c_1+c_2)^2}{4c_1c_2} < \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}
\end{aligned}$$

The third condition is always satisfied for $c_1 = c_2 = c$, whereas the equilibrium may become unstable through a Neimark-Sacker bifurcation when $\frac{c_1}{c_2} \neq 1$. In particular, without any inertia, i.e. $\lambda_1 = \lambda_2 = 1$, the unique positive equilibrium is stable if and only if $c_1/c_2 \in (3-2\sqrt{2}, 3+2\sqrt{2})$.

It is worth to notice that the map (22) is a particular case of (6) because after the change of coordinates

$$x = c_2 q_1 \ ;\ y = c_1 q_2 \tag{24}$$

the following map is obtained

$$\begin{aligned}
x' &= (1-\lambda_1)x + a\lambda_1\left(\sqrt{y} - y\right) \\
y' &= (1-\lambda_2)y + \tfrac{1}{a}\lambda_2\left(\sqrt{x} - x\right)
\end{aligned}$$

where $a = c_2/c_1$. This is a particular case of (6) with $a_1 = a$ and $a_2 = \frac{1}{a}$, and this explains why the creation of further fixed points is not possible, as explained in the remark after Proposition 2, Indeed, the unique equilibrium $E = \left(\frac{1}{(a_1+1)^2}, \frac{1}{(a_2+1)^2}\right)$ existing in the particular case $a_1a_2 = 1$ corresponds to $E_1 = \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right)$ by considering $a_1 = c_2/c_1$, $a_2 = c_1/c_2$, together with (24). We refer to [1] and [2] for a deeper analysis of the same model.

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